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Acceleration-enlarged symmetries in nonrelativistic space-time with a cosmological constant[a]

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Abstract. By considering the nonrelativistic limit of de Sitter geometry one obtains the nonrelativistic space-time with a cosmological constant and Newton–Hooke (NH) symmetries. We show that the NH symmetry algebra can be enlarged by the addition of the constant acceleration generators and endowed with central extensions (one in any dimension (D) and three in $D=(2+1)$). We present a classical Lagrangian and Hamiltonian framework for constructing models quasi-invariant under enlarged NH symmetries that depend on three parameters described by three nonvanishing central charges. The Hamiltonian dynamics then splits into external and internal sectors with new noncommutative structures of external and internal phase spaces. We show that in the limit of vanishing cosmological constant the system reduces to the one, which possesses acceleration-enlarged Galilean symmetries.

1 Introduction

The two nonrelativistic Newton–Hooke (NH) cosmological groups were introduced by Bacry and Leblond in [2] who classified all kinematical groups in $D=(3+1)$. The NH symmetries can be obtained, in the limit $c\to\infty$, from the de Sitter and anti-de Sitter geometries [2–7], and they describe, respectively, the nonrelativistic expanding (with symmetry described by the NH_+ algebra) and oscillating (with symmetry described by the NH_- algebra) universes. The cosmological constant describes the time scale τ determining the rate of expansion or the period of oscillation of the universe. When $\tau\to\infty$ we obtain the Galilean group and the standard flat nonrelativistic space-time. It has been argued that the NH symmetries and the corresponding NH space-times can find an application in nonrelativistic cosmology [8–10] or even in M-theory and string theory [11].

Recently, we have considered in [1] the acceleration-enlarged Galilean symmetries[1] with their central extensions and their dynamical realisations. The aim of this note is to show that these results, corresponding to $\tau\to\infty$, can be generalised to the NH spaces with a finite cosmological constant (τ finite). In particular, we show that the acceleration-enlarged NH symmetries, as in the Galilean case, can have

1. for arbitrary D one central extension,
2. for $D=(2+1)$ three central extensions.

We also generalise to $\tau<\infty$ the actions providing the dynamical realisations of the NH algebra and the corresponding equations of motion.

The plan of our presentation is as follows. First, in Sect. 2 we recall some known results [5–7, 9, 12] on the NH algebras with central extensions. We also add the constant acceleration generators, consider the most general central extensions and describe the Casimir operators of our new algebras. In Sect. 3 we present a $(2+1)$-dimensional classical mechanics Lagrangian model with higher derivatives providing a dynamical realisation of the introduced symmetries. In Sect. 4 we consider its phase space formulation – with five 2-vector coordinates (x_i,p_i,v_i,q_i,u_i). In the following section we decompose its dynamics into two sectors ("external" and "internal"), with the six-dimensional external sector describing a new $D=2$ noncommutative extended phase space. The last section contains some final remarks.

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[1] In [1] we described our procedure of adding constant acceleration generators as an 'extension'. To avoid confusion with the mathematical terminology (G' is an extension of G by K if $G=\frac{G'}{K}$) we call, in this paper, our procedure an 'enlargement'.

2 Acceleration-enlarged Newton–Hooke symmetries with central charges – algebraic considerations

Let us first recall the NH_+ algebra – corresponding to an expanding universe. In this case the Galilean algebra with the following nonvanishing commutators ($i,j,k,l = 1,\ldots,D-1$):

$$[J_{ij}, J_{kl}] = \delta_{ik}J_{jl} - \delta_{jl}J_{ik} + \delta_{jk}J_{il} - \delta_{il}J_{jk}\,, \tag{1a}$$

$$[J_{ij}, A_k] = \delta_{ik}A_j - \delta_{jk}A_i \quad (A_i = P_i, K_i)\,, \tag{1b}$$

$$[H, K_i] = P_i\,, \tag{1c}$$

is supplemented by the additional commutators[2]

$$[H, P_i] = \frac{K_i}{R^2} \tag{1d}$$

obtained by the deformation of the Galilean relations $[H, P_i] = 0$ [13].

In an arbitrary number of dimensions D, as in the Galilean case, one can introduce a central extension describing the mass m[3]

$$[P_i, K_j] = m\delta_{ij}. \tag{2}$$

In $D = (2+1)$ one can have a second central charge θ [7, 9, 12]

$$[K_i, K_j] = \theta\epsilon_{ij}\,. \tag{3a}$$

Algebraic consistency then requires that

$$[P_i, P_j] = -\frac{\theta}{R^2}\epsilon_{ij}\,. \tag{3b}$$

The NH algebra can be enlarged to the algebra $\widehat{\mathrm{NH}}$ by the addition of the generators F_i that describe the constant acceleration transformations [5, 6]. These generators satisfy the relation (1b) and

$$[F_i, F_j] = [F_i, K_j] = 0, \quad [H, F_i] = 2K_i\,. \tag{4}$$

One can show that in an arbitrary space-time dimension D one can introduce one central charge c as follows:

$$[K_i, F_j] = 2c\delta_{ij}\,. \tag{5}$$

The algebraic consistency then requires that

$$[P_i, K_j] = -\frac{c}{R^2}\delta_{ij}\,, \tag{6}$$

i.e. we obtain the R-dependent mass parameter, vanishing in the limit $R \to \infty$.

In $D = (2+1)$ one can introduce, besides c, two additional central charges θ and θ'. The first one is already present in (3) and it appears in the following central charge commutator [14]:

$$[P_i, F_j] = -2\theta\epsilon_{ij}\,. \tag{7}$$

The second central charge occurs only in the relation

$$[F_i, F_j] = \theta'\epsilon_{ij}\,. \tag{8}$$

However, it is easy to check that θ is a genuine central charge, determining the structure of the enveloping algebra; the other two can be generated by the linear transformations of the generators (P_i, K_i, F_i) in the enveloping algebra $U(\widehat{\mathrm{NH}}_\theta)$.[4] We have (see [1] and cp [13])

$$\begin{aligned}
P_i \to \tilde{P}_i &= \gamma P_i + \frac{c}{2\theta R^2\gamma}\epsilon_{ij}K_j\,,\\
K_i \to \tilde{K}_i &= \gamma K_i + \frac{c}{2\theta\gamma}\epsilon_{ij}P_j\,,\\
F_i \to \tilde{F}_i &= \left(\gamma - \frac{\rho}{2R^2}\right)F_i + \frac{c}{\gamma\theta}\epsilon_{ij}K_j + \rho P_i\,,
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
\gamma^2 &= \frac{1}{2}\left[1 + \left(1 + \left(\frac{c}{\theta R}\right)^2\right)^{\frac{1}{2}}\right],\\
\rho &= 2\gamma R^2\left[1 - \left(1 + \frac{\theta' - \frac{c^2}{\theta\gamma}}{4\gamma^2\theta R^2}\right)^{\frac{1}{2}}\right].
\end{aligned} \tag{10}$$

Note that as $R \to \infty$ we have

$$\gamma \to 1\,, \quad \rho \to \frac{c^2}{4\theta^2} - \frac{\theta'}{4\theta}\,, \tag{11}$$

and so we see that (9) and (10) generalise to the case $R < \infty$ the relevant expressions given in [1].

Finally, let us mention the two Casimir operators of the acceleration-enlarged Newton–Hooke algebra with central charges. They are

$$C_H = H + \frac{c}{2\theta^2}\frac{1}{1 + \frac{c^2}{\theta^2R^2}}\left(\vec{P}^2 - \frac{\vec{K}^2}{R^2}\right) - \frac{1}{\theta\left(1 + \frac{c^2}{\theta^2R^2}\right)}\epsilon_{ij}K_iP_j \tag{12a}$$

and, putting $J_{ij} = \epsilon_{ij}J$ for $D = (2+1)$

$$C_J = J - a\left(\vec{F}\vec{P} - \vec{K}^2 - \frac{\vec{F}^2}{4R^2}\right) - \frac{c}{\theta}H - b\left(\vec{P}^2 - \frac{\vec{K}^2}{R^2}\right), \tag{12b}$$

where

$$a = \frac{1}{2\theta + \frac{\theta'}{2R^2}}\,, \quad b = \frac{a\theta'}{4\theta}\,. \tag{12c}$$

[2] We pass to the NH_- algebra by the substitution $\frac{1}{R^2} \to -\frac{1}{R^2}$ in (1d).

[3] In this paper we represent our central generators as $\lambda\mathbf{1}$, where λ is a number, and we omit the unity operator $\mathbf{1}$.

[4] By $\widehat{\mathrm{NH}}_\theta$ we denote the acceleration-enlarged NH algebra with one central charge θ.

One can check that in the limit $R \to \infty$ these Casimir operators reduce to those of the acceleration-enlarged Galilei algebra given in [1].

Note that in $D=(2+1)$ we can add to the above given $\widehat{\mathrm{NH}}_+$ algebra with one central charge two further generators

$$J_\pm = \frac{1}{4\theta}\left(K_\pm^2 - F_\pm P_\pm + \frac{F_\pm^2}{4R^2}\right), \tag{13}$$

where $K_\pm = K_1 \pm \mathrm{i}K_2$ etc. If we now put $C_J = 0$ and express J from (12b), the set of three generators (J, J_+, J_-) provides a basis of a Lie algebra $\mathrm{O}(2,1)$. If we now make the substitution $R^2 \to -R^2$ (i.e. $(\widehat{\mathrm{NH}}_+ \to \widehat{\mathrm{NH}}_-)$, as can be easily checked this algebra becomes $\mathrm{O}(3)$. In fact, an analogous result for the exotic ((2+1)-dimensional case with two central charges) NH algebras, in the case of $F_i = 0$, was discussed in [12].

3 $D=(2+1)$ Lagrangian models with acceleration-enlarged NH symmetries and with central charges

When we put central charges to zero our algebra given by (1) and (4) can be realised by the following differential operators on the nonrelativistic $D=(2+1)$ space-time (x_i, t) $(i=1,2)$:[5]

$$\begin{aligned} H &= \frac{\partial}{\partial t}, \qquad & J &= \epsilon_{ij} x_i \frac{\partial}{\partial x_j}, \\ P_i &= \cosh\frac{t}{R}\frac{\partial}{\partial x_i}, \qquad & K_i &= R\sinh\frac{t}{R}\frac{\partial}{\partial x_i}, \\ F_i &= 2R^2\left(\cosh\frac{t}{R} - 1\right)\frac{\partial}{\partial x_i}. \end{aligned} \tag{14}$$

Note that we have the following $\widehat{\mathrm{NH}}$ transformations of the $D=(2+1)$ space-time (δa_i is for translations, δv_i for boosts, δb_i for constant accelerations, δa for time displacement and $\delta\alpha$ for $\mathrm{O}(2)$ angle rotations):

$$\begin{aligned} \delta t &= \delta a, \\ \delta x_i &= \delta\alpha \epsilon_{ij} x_j + \cosh\frac{t}{R}\delta a_i \\ &\quad + R\sinh\frac{t}{R}\delta v_i + 2R^2\left(\cosh\frac{t}{R} - 1\right)\delta b_i. \end{aligned} \tag{15}$$

Next we look for Lagrangians that are quasi-invariant under the transformations (15) and that, in the limit $R \to \infty$, reduce to the higher derivative Lagrangian given in [1], namely

$$L_{R=\infty}(c,\theta,\theta') = -\frac{\theta}{2}\epsilon_{ij}\dot{x}_i\ddot{x}_j + \frac{c}{2}\ddot{x}_i^2 - \frac{\theta'}{8}\epsilon_{ij}\ddot{x}_i\dddot{x}_j. \tag{16}$$

Note that when $c = \theta' = 0$ and adding, for $m \neq 0$, the kinetic term $\frac{m}{2}\dot{x}_i^2$, we get the Lagrangian considered in [15].

[5] Note that for finite R suitable linear combinations of P_i and F_i generate standard translations.

The extension of this Lagrangian to the NH case can be obtained via the substitution (see e.g. [9, 12]):

$$\epsilon_{ij}\dot{x}_i\ddot{x}_j \to \epsilon_{ij}\left(\dot{x}_i\ddot{x}_j + \frac{1}{R^2}x_i\dot{x}_j\right). \tag{17}$$

It is easy to check that this substitution can also be generalised to the other two terms in (16) as follows:

$$\begin{aligned} \ddot{x}_i^2 &\to \ddot{x}_i^2 + \frac{1}{R^2}\dot{x}_i^2, \\ \epsilon_{ij}\ddot{x}_i\dddot{x}_j &\to \epsilon_{ij}\left(\ddot{x}_i\dddot{x}_j + \frac{1}{R^2}\dot{x}_i\ddot{x}_j\right). \end{aligned} \tag{18}$$

It is easy to check that the RHS terms in (17) and (18) are quasi-invariant under the transformations (15). If we perform the substitutions (18) together with $\theta \to \tilde{\theta} = \theta + \frac{\theta'}{4R^2}$, we obtain

$$\begin{aligned} L_{\mathrm{NH}} = &-\frac{\tilde{\theta}}{2}\epsilon_{ij}\left(\dot{x}_i\ddot{x}_j + \frac{1}{R^2}x_i\dot{x}_j\right) + \frac{c}{2}\left(\ddot{x}_i^2 + \frac{1}{R^2}\dot{x}_i^2\right) \\ &- \frac{\theta'}{8}\epsilon_{ij}\left(\ddot{x}_i\dddot{x}_j + \frac{1}{R^2}\dot{x}_i\ddot{x}_j\right). \end{aligned} \tag{19}$$

Note that the substitution $\theta \to \tilde{\theta}$ is necessary to get agreement with the definition of the central charge θ in (3a).

When, below, we rewrite our Lagrangian (19) in the first order Hamiltonian formalism we need to introduce five pairs of variables $(x_i, p_i, y_i, q_i, u_i)$.

4 First order Hamiltonian formalism and the extended phase space

First we define two new 2-vector variables by

$$y_i = \dot{x}_i, \quad u_i = \dot{y}_i. \tag{20}$$

Then we obtain the following first order Lagrangian[6]:

$$\begin{aligned} L_{\mathrm{NH}} = &\, p_i(\dot{x}_i - y_i) + q_i(\dot{y}_i - u_i) - \frac{\tilde{\theta}}{2}\epsilon_{ij}y_i\dot{y}_j - \frac{\theta'}{8}\epsilon_{ij}u_i\dot{u}_j \\ &+ \frac{c}{2}\left(u_i^2 + \frac{y_i^2}{R^2}\right) - \frac{\tilde{\theta}}{2R^2}\epsilon_{ij}x_i y_j - \frac{\theta'}{8R^2}\epsilon_{ij}y_i u_j, \end{aligned} \tag{21}$$

which, after the substitution (20), is equivalent to the higher order action (19). Using the Faddeev–Jackiw prescription [17, 18] we obtain the following nonvanishing Poisson brackets:

$$\begin{aligned} \{x_i, p_j\} &= \delta_{ij}, \qquad & \{y_i, q_j\} &= \delta_{ij}, \\ \{q_i, q_j\} &= -\tilde{\theta}\epsilon_{ij}, \qquad & \{u_i, u_j\} &= \frac{4}{\theta'}\epsilon_{ij}. \end{aligned} \tag{22}$$

[6] The first order quasi-invariant actions can be also derived by a geometric technique that involves the method of nonlinear realisations of the $\widehat{\mathrm{NH}}_+$ group with central extensions and the inverse Higgs mechanism (for the NH_+ group see [12]; see also [16]). It would be interesting to describe the most general class of first order actions that can be generated in this way.

The Hamiltonian, which follows from (21), has the form

$$H = p_i y_i + q_i u_i - \frac{c}{2}\left(u_i^2 + \frac{1}{R^2} y_i^2\right) + \frac{\tilde{\theta}}{2R^2}\epsilon_{ij} x_i y_j + \frac{\theta'}{8R^2}\epsilon_{ij} y_i u_j \,. \tag{23}$$

The equations of motion, which can be obtained from (21) as the Euler–Lagrange equations or as the Hamilton equations following from (22) and (23), give, besides (20), also

$$\dot{u}_i = \frac{1}{2R^2} y_i + \frac{4}{\theta'}\epsilon_{ij}(q_j - c y_j)\,, \tag{24a}$$

$$\dot{q}_i = -\left(\tilde{\theta} + \frac{\theta'}{8R^2}\right)\epsilon_{ij} u_j + \frac{\tilde{\theta}}{2R^2}\epsilon_{ij} x_j + \frac{c}{R^2} y_i - p_i\,, \tag{24b}$$

$$\dot{p}_i = -\frac{\tilde{\theta}}{2R^2}\epsilon_{ij} y_j\,. \tag{24c}$$

Using the first two equations as defining the variables q_j, p_i we obtain the following set of the $\widehat{\mathrm{NH}}$ transformation laws for the extended phase space coordinates $(x_i, p_i, y_i, q_i, u_i)$:

- $\widehat{\mathrm{NH}}_+$ translations (parameters a_i)

$$\delta x_i = a_i \cosh\frac{t}{R}\,, \qquad y_i = \frac{a_i}{R}\sinh\frac{t}{R}\,,$$
$$\delta u_i = \frac{a_i}{R^2}\cosh\frac{t}{R}\,, \qquad \delta p_i = -\epsilon_{ij} a_j \frac{\tilde{\theta}}{2R^2}\cosh\frac{t}{R}\,,$$
$$\delta q_i = a_i \frac{c}{R^2}\cosh\frac{t}{R} - \epsilon_{ij} a_j \frac{\theta'}{8R^3}\sinh\frac{t}{R}\,; \tag{25}$$

- $\widehat{\mathrm{NH}}_+$ boosts (parameters v_i)

$$\delta x_i = v_i R \sinh\frac{t}{R}\,, \qquad \delta y_i = v_i \cosh\frac{t}{R}\,,$$
$$\delta u_i = \frac{v_i}{R}\sinh\frac{t}{R}\,, \qquad \delta p_i = -\epsilon_{ij} v_j \frac{\tilde{\theta}}{2R}\sinh\frac{t}{R}\,,$$
$$\delta q_i = \frac{v_i c}{R}\sinh\frac{t}{R} - \epsilon_{ij} v_j \frac{\theta'}{8R^2}\cosh\frac{t}{R}\,; \tag{26}$$

- $\widehat{\mathrm{NH}}_+$ accelerations (parameters b_i)

$$\delta x_i = 2b_i R^2\left(\cosh\frac{t}{R} - 1\right), \quad \delta y_i = 2b_i R \sinh\frac{t}{R}\,,$$
$$\delta u_i = 2b_i \cosh\frac{t}{R}, \qquad p_i = -\epsilon_{ij} b_j \tilde{\theta}\left(\cosh\frac{t}{R} + 1\right),$$
$$\delta q_i = 2b_i c \cosh\frac{t}{R} - \epsilon_{ij} b_j \frac{\theta'}{4R^2}\sinh\frac{t}{R}\,. \tag{27}$$

We leave it as an exercise for the interested reader to determine the Noether charges generating (25)–(27). They will satisfy the acceleration-enlarged $\widehat{\mathrm{NH}}$ algebra with central charges introduced in Sect. 2.

5 "External" and "internal" dynamics sectors with noncommutative structure

Next we look for internal variables that are invariant under the transformations (25)–(27) and so could be used to parametrise the internal sector of the phase space with the time evolution being an internal automorphism. We note that for these internal sector variables we can take

$$U_i = u_i - \frac{1}{\tilde{\theta}}\epsilon_{ij}\left(p_j + \frac{\tilde{\theta}}{2R^2}\epsilon_{jk} x_k\right) - \frac{x_i}{R^2}\,,$$
$$Q_i = q_i - c u_i + \frac{\theta'}{8R^2}\epsilon_{ij} y_j\,, \tag{28}$$

as they are invariant under the transformations (25)–(27).

From (20) and (24) we get

$$\dot{U}_i = \frac{4}{\theta'}\epsilon_{ij} Q_j\,,$$
$$\dot{Q}_i = -\frac{4c}{\theta'}\epsilon_{ij} Q_j - \tilde{\theta}\epsilon_{ij} U_j\,. \tag{29}$$

Hence we see that the dynamics of the variables U_i and Q_i is closed under the time evolution and, as such, it describes the "internal" dynamics in our model. Let us note the following:

- The set of equations (29) leads to the fourth order equations for the individual components of the fields U_i and Q_i.
- The description of the dynamics of the internal sector (U_i, Q_i) does not depend on R, i.e. it is the same in the Galilean limit ($R \to \infty$) considered in [1] after the substitution $\theta \to \tilde{\theta}$ has been made.

To describe the 'external' sector of our model we have to introduce three sets of variables (X_i, P_i, Y_i) that have vanishing Poisson brackets with U_i and Q_i. Then the two sectors will be dynamically independent.

Note that from (28) and (22) we have

$$\{U_i, U_j\} = 4\frac{\theta}{\theta'\tilde{\theta}}\epsilon_{ij}\,, \qquad \{Q_i, U_j\} = -\frac{4c}{\theta'}\epsilon_{ij}\,,$$
$$\{Q_i, Q_j\} = \epsilon_{ij}\left(-\tilde{\theta} + \frac{4c^2}{\theta'} + \frac{\theta'}{4R^2}\right). \tag{30}$$

Then, if we define

$$X_i = x_i + \alpha Q_i + \beta U_i\,, \tag{31}$$

we find that

$$\{X_i, Q_j\} = \{X_i, U_j\} = 0 \tag{32}$$

are satisfied if

$$\alpha = \left(\frac{\theta^2}{c} + \frac{c}{R^2}\right)^{-1}, \quad \beta = \left(c - \frac{\theta\theta'}{4c}\right)\alpha\,. \tag{33}$$

For Y_i we try the ansatz $Y_i = \dot{X}_i$. Then using the Jacobi identity and the field equations (29), we see that[7]

$$\{Y_i, U_j\} = \{\{X_i, H\}, U_j\} = -\{X_i, \{Q_j, H\}\} = -\frac{4}{\theta'}\epsilon_{ij}\{X_i, Q_j\} = 0\,. \quad (34)$$

Thus we see that we can put $Y_i = \dot{X}_i$, or we can use the field equations (30) and note that

$$Y_i = y_i - \alpha\left(\tilde{\theta}\epsilon_{ij}U_j + \frac{\theta}{c}\epsilon_{ij}Q_j\right). \quad (35)$$

To find the third set of variables of the external variables we calculate the time derivative of Y_i. We find

$$\dot{Y}_i = \frac{X_i}{R^2} + \frac{1}{\tilde{\theta}}\epsilon_{ij}P_j\,, \quad (36)$$

where P_i is given by

$$P_i = p_i + \frac{\tilde{\theta}}{2R^2}\epsilon_{ij}x_j\,. \quad (37)$$

Note the following:

- The variables P_i are invariant under the $\widehat{\mathrm{NH}}_+$ translations and $\widehat{\mathrm{NH}}_+$ boosts.
- The "external" 2-momenta P_i are constant "on-shell", i.e. from (37), (20), and (24c) we see that

$$\dot{P}_i = 0\,. \quad (38)$$

The equal time noncommutative structure of the "external" sector variables $X_i, P_i, Y_i = \dot{X}_i$ is determined by the following set of Poisson brackets:

$$\{X_i, P_j\} = \delta_{ij}\,, \qquad \{P_i, P_j\} = \frac{\tilde{\theta}}{R^2}\epsilon_{ij}\,,$$
$$\{X_i, X_j\} = \frac{\beta}{\tilde{\theta}}\epsilon_{ij}\,, \qquad \{X_i, Y_j\} = -\alpha\frac{\theta}{c}\epsilon_{ij}\,,$$
$$\{Y_i, Y_j\} = \alpha\frac{\theta}{c}\epsilon_{ij}\,, \qquad \{Y_i, P_j\} = 0\,. \quad (39)$$

Therefore, by means of the substitution $X_i \to X'_i = X_i + \frac{R^2}{\tilde{\theta}}\epsilon_{ik}P_k$ we obtain the equations of the hyperbolic oscillator:

$$\ddot{X}'_i = \frac{1}{R^2}X'_i \quad (40)$$

and a decoupling of P_i from (X'_i, Y_i) in the Poisson brackets (39).

We see that our present model differs in many respects from the original higher derivative model [15, 19].

- The external sector, described in [15] by a pair (X_i, P_i) of noncommutative X_i and commutative P_i variables, is replaced now by a triplet of variables (X_i, P_i, Y_i) with both X_i and P_i being noncommutative (see (39)).
- The internal sector described in [15] by a single 2-vector is spanned in the present model by a pair of 2-vectors (Q_i, U_i) with the dynamics described by the equations with the fourth order time derivatives (see e.g. [20, 21]). However, after a complex point transformation $(Q_i, U_i) \to (z_i, \pi_i)$ the internal part L_{int} of the first order Lagrangian (20) can be put into the standard form with a canonical symplectic structure ($L_{\mathrm{int}} = \dot{z}_i\pi - H_{\mathrm{int}}(z_i, \pi_i)$) [22].

[7] In a similar way we can show that $\{Y_i, Q_j\} = 0$.

6 Final remarks

This paper has had three aims.

- To show that the acceleration-enlarged Galilean symmetries can be generalised to the acceleration-enlarged Newton–Hooke symmetries describing very special nonrelativisitic de Sitter geometries. In particular, we have shown that the acceleration-enlarged $\widehat{\mathrm{NH}}$ algebra in $(D = 2+1)$ dimensions has three central charges.
- It has been known for some time that mass, as a central charge of the Galilean symmetry, provides a unique parameter that controls the free particle motion. Now we see that in $D = (2+1)$ dimensions the $\widehat{\mathrm{NH}}$ invariant action depends on three parameters that are determined by three central charges of the acceleration-enlarged $\widehat{\mathrm{NH}}$ algebra.
- It was shown in [15] that in $D = (2+1)$ dimensions the exotic central charge of the Galilean algebra generates the noncommutativity of the $D = 2$ nonrelativistic space coordinates. In this paper we have shown that for a system with $\widehat{\mathrm{NH}}$ symmetries (dS radius R finite) one can generate a more complicated nonrelativistic $D = 2$ phase space with both coordinates and momenta being noncommutative. We have found also that the complete noncommutative phase space describing the external sector of our model, besides (X_i, P_i), possesses also a third noncommutative 2-vector.

As is well known the quantization of models with higher time derivatives leads to the canonical derivation of deformed quantum phase space structures. It still remains to be clarified how the well known difficulty with the quantization of mechanical systems with higher order equations of motion, leading to the appearance of ghosts, can be remedied if we formulate the model in the framework of suitably extended and deformed quantum mechanics.

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